TOTAL EDGE IRREGULARITY STRENGTH OF SOME FAMILIES OF GRAPHS

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Abstract

An edge irregular total k-labeling $f : V \cup E \to \{1, 2, 3, \dots, k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of G in such a way that for any two different edges uv and $u'v'$ their weights $f(u) + f(uv) + f(v)$ and $f(u') + f(u'v') + f(v')$ are distinct. The total edge irregularity strength $tes(G)$ is defined as the minimum k for which the graph G has an edge irregular total k-labeling. In this paper, we study the total edge irregularity strength for tadpole $T(n, t)$, $n \geq 3$, $t \geq 1$, armed crown $(C_n \ominus P_m)$, for $n \geq 3, m \geq 1$, split graph of cycle and split graph of a path.

Keywords: Irregularity strength, total edge irregularity strength, edge irregular total labeling, split graph.
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1 Introduction

All the graphs considered in this paper are simple, finite and undirected. In [11] *Bača* et.al. introduced the notion of an edge irregular total k-labeling of a graph G as a function $\phi : V \cup E \to \{1, 2, 3, \dots, k\}$ such that the edge weights$wt_\phi(uv) = \phi(u) + \phi(uv) + \phi(v)$ are distinct for all edges. That is $wt_\phi(uv) \neq wt_\phi(u'v')$ for every pair of edges $uv, u'v' \in E$. The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength of G, denoted by $tes(G)$. Further *Bača* et.al. found a lower bound for the total edge irregularity strength of any graph as

$$tes(G) \geq max\left\{ \left\lceil \frac{(|E(G)| + 2)}{3} \right\rceil, \left\lceil \frac{(\Delta(G) + 1)}{2} \right\rceil \right\} \tag{1}$$

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where $\Delta(G)$ is the maximum degree of G. In [13] *Ivančo* and *Jendroľ*, posed the following conjecture:

Conjecture 1.1. [13] *Let G be an arbitrary graph different from K_5. Then*

$$tes(G) = max\left\{\left\lceil\frac{(|E(G)| + 2)}{3}\right\rceil, \left\lceil\frac{(\Delta(G) + 1)}{2}\right\rceil\right\} \tag{2}$$

Conjecture (1.1) has been verified for the categorical product of a cycle [1], strong product [2], categorical product of two paths $P_n \times P_m$ [3], categorical product of two cycles[4], strong product of cycles and paths[5], disjoint union of prisms and cycles[6], zigzag graphs[7], hexagonal grid graphs[8], generalized prism [9], toroidal fullerene[10], large dense graphs with $\frac{(|E(G)|+2)}{3} \leq \frac{(\Delta(G)+1)}{2}$ [12], trees [13], complete graphs [14], complete bipartite graphs [15], disjoint union of wheel graphs [16], cartesian product of two paths $P_n \times P_m$ [17], corona product of a path with certain graphs [18], categorical product of cycle and path [19], and for subdivision of star [20]. In this paper, we determine exact values of the total edge irregularity strength for tadpole $T(n,t)$, $n \geq 3$, $t \geq 1$, armed crown $(C_n \ominus P_m)$, $n \geq 3$, $m \geq 1$, split graph of cycle and split graph of path.

Definition 1.2. *An (n,t)-kite is a cycle of length n with a t-edge path (the tail) attached to one vertex and it is also called tadpole $T(n,t)$ or dragon graph.*

Definition 1.3. *The armed crown $C_n \ominus P_m$ is a graph in which the path P_m is attached at each vertex of cycle C_n by an edge. Thus the vertex set is $V(C_n \ominus P_m) = \{v_i, p^i_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set is $E(C_n \ominus P_m) = \{v_iv_{i+1}, p^i_m v_i, p^i_j p^i_{j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$, with indices taken modulo n. When $m = 1$ this graph is called sun graph.*

Definition 1.4. *The split graph $spl(G)$ of a graph G is obtained by adding a new vertex v' to each vertex v such that v' is adjacent to every vertex which is adjacent to v in G.*

2 Main Results

Theorem 2.1. $tes(T(n,t)) = \left\lceil\frac{n+t+2}{3}\right\rceil$, $n \geq 3$, $t \geq 1$.

Proof. Let $V(T(n,t)) = \{v_i, 1 \leq i \leq n\} \cup \{u_i, 1 \leq i \leq t\}$ and
$E(T(n,t)) = \{u_iu_{i+1}, 1 \leq i \leq t-1\} \cup \{v_iv_{i+1}, v_su_1, 1 \leq i \leq n, s \in \{1,2,3,\ldots,n\}\}$ with indices taken modulo n. Let $k = \left\lceil\frac{n+t+2}{3}\right\rceil$. Then from (1) it follows that,
$tes(T(n,t)) \geq \left\lceil\frac{|E(T(n,t))|+2}{3}\right\rceil = \left\lceil\frac{n+t+2}{3}\right\rceil = k$.
To prove the reverse inequality, we define a function f from $V \cup E$ to $\{1,2,3,\ldots,k\}$ by considering the following three cases.

Case (i): $\left\lceil \frac{n}{2} \right\rceil + 1 > k$.

$$f(v_i) = \begin{cases} \left\lceil \frac{i}{2} \right\rceil, & if\ 1 \le i \le 2k-2 \\ k, & if\ 2k-1 \le i \le n; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 2+i-\left\lceil \frac{i}{2} \right\rceil - \left\lceil \frac{i+1}{2} \right\rceil, & if\ 1 \le i \le 2k-2 \\ 3+i-2k, & if\ 2k-1 \le i \le n-1 \\ k, & if\ i=n; \end{cases}$$

$f(u_i) = k,\ 1 \le i \le t$; $f(u_iu_{i+1}) = n+3-2k+i,\ 1 \le i \le t-1$;
$f(v_su_1) = n+3-2k,\ if\ s=n$.
We observe that,

$$wt(v_iv_{i+1}) = \begin{cases} 2+i, & if\ 1 \le i \le 2k-2 \\ 3+i, & if\ 2k-1 \le i \le n-1 \\ 2k+1, & if\ i=n; \end{cases}$$

$wt(v_su_1) = n+3$, where $s=n$; $wt(u_iu_{i+1}) = n+3+i,\ 1 \le i \le t-1$.
Case (ii): $\left\lceil \frac{n}{2} \right\rceil + 1 = k$.
$f(v_i) = \left\lceil \frac{i}{2} \right\rceil,\ 1 \le i \le n$;

$$f(v_iv_{i+1}) = \begin{cases} 2+i-\left\lceil \frac{i}{2} \right\rceil - \left\lceil \frac{i+1}{2} \right\rceil, & if\ 1 \le i \le n-1 \\ n+1-\left\lceil \frac{n}{2} \right\rceil, & if\ i=n; \end{cases}$$

$f(u_i) = k,\ 1 \le i \le t$; $f(u_iu_{i+1}) = n+3-2k+i,\ 1 \le i \le t-1$;
$f(v_su_1) = n+3-k-\left\lceil \frac{s}{2} \right\rceil,\ s \in \{1,2,3,\ldots,n\}$.
We observe that,

$$wt(v_iv_{i+1}) = \begin{cases} 2+i, & if\ 1 \le i \le n-1 \\ n+2, & if\ i=n; \end{cases}$$

$wt(v_su_1) = n+3,\ s \in \{1,2,3,\ldots,n\}$; $wt(u_iu_{i+1}) = n+3+i,\ 1 \le i \le t-1$.
Case (iii): $\left\lceil \frac{n}{2} \right\rceil + 1 < k$.
This case contains four subcases:
$f(v_i) = \left\lceil \frac{i}{2} \right\rceil,\ 1 \le i \le n$;

$$f(v_iv_{i+1}) = \begin{cases} 2+i-\left\lceil \frac{i}{2} \right\rceil - \left\lceil \frac{i+1}{2} \right\rceil, & if\ 1 \le i \le n-1 \\ n+1-\left\lceil \frac{n}{2} \right\rceil, & if\ i=n; \end{cases}$$

$f(v_su_1) = n+2-\left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{s}{2} \right\rceil,\ s \in \{1,2,3,\ldots,n\}$.
Subcase (1): n is even and $t < 2n+8$.

$$f(u_i)=\begin{cases}\frac{n+2}{2}, & if\ 1\le i\le k\\ k, & if\ k+1\le i\le t;\end{cases}$$

$$f(u_iu_{i+1})=\begin{cases}1+i, & if\ 1\le i\le k-1\\ \frac{n+4}{2}, & if\ i=k\\ n+3-2k+i, & if\ k+1\le i\le t-1.\end{cases}$$

We observe that,

$$wt(v_iv_{i+1})=\begin{cases}2+i, & if\ 1\le i\le n-1\\ n+2, & if\ i=n;\end{cases}$$

$$wt(u_iu_{i+1})=\begin{cases}n+3+i, & if\ 1\le i\le k-1\\ n+k+3, & if\ i=k\\ n+3+i, & if\ k+1\le i\le t-1;\end{cases}$$

$wt(v_su_1)=n+3,\ s\in\{1,2,3,\dots,n\}$.

Subcase (2): n is even and $t\ge 2n+8$.

$$f(u_i)=\begin{cases}\left\lceil\frac{i}{2}\right\rceil, & if\ 1\le i\le 2k-2\\ k, & if\ 2k-1\le i\le t;\end{cases}$$

$f(v_i)=k, 1\le i\le n;$

$$f(u_iu_{i+1})=\begin{cases}2-\left\lceil\frac{i}{2}\right\rceil-\left\lceil\frac{i+1}{2}\right\rceil+i, & if\ 1\le i\le 2k-2\\ 3-2k+i, & if\ 2k-1\le i\le t-1;\end{cases}$$

$f(v_iv_{i+1})=t+2-2k+i,\ if 1\le i\le n;\ f(v_su_1)=k,\ s\in\{1,2,3,\dots,n\}$.

We observe that, $wt(v_iv_{i+1})=t+2+i,\ 1\le i\le n;\ wt(v_su_1)=2k+1,\ s\in\{1,2,3,\dots,n\}$;

$$wt(u_iu_{i+1})=\begin{cases}2+i, & if\ 1\le i\le 2k-2\\ 3+i, & if\ 2k-1\le i\le t-1.\end{cases}$$

Subcase (3): n is odd and $t<2n+11$.

$$f(u_i)=\begin{cases}\frac{n+3}{2}, & if\ 1\le i\le k+1\\ k, & if\ k+2\le i\le t;\end{cases}$$

$$f(u_iu_{i+1})=\begin{cases}i, & if\ 1\le i\le k\\ \frac{n+5}{2}, & if\ i=k+1\\ n+3-2k+i, & if\ k+2\le i\le t-1.\end{cases}$$

We observe that,

$$wt(v_iv_{i+1}) = \begin{cases} 2+i, & if \ 1 \le i \le n-1 \\ n+2, & if \ i = n; \end{cases}$$

$$wt(u_iu_{i+1}) = \begin{cases} n+3+i, & if \ 1 \le i \le k \\ n+k+4, & if \ i = k+1 \\ n+3+i, & if \ k+2 \le i \le t-1; \end{cases}$$

$wt(v_su_1) = n+3, \ s \in \{1,2,3,\ldots,n\}$.

Subcase (4): n is odd and $t \ge 2n+11$.

$$f(u_i) = \begin{cases} \left\lceil \frac{i}{2} \right\rceil, & if \ 1 \le i \le 2k-2 \\ k, & if \ 2k-1 \le i \le t; \end{cases}$$

$f(v_i) = k, 1 \le i \le n;$

$$f(u_iu_{i+1}) = \begin{cases} 2 - \left\lceil \frac{i}{2} \right\rceil - \left\lceil \frac{i+1}{2} \right\rceil + i, & if \ 1 \le i \le 2k-2 \\ 3-2k+i, & if \ 2k-1 \le i \le t-1; \end{cases}$$

$f(v_iv_{i+1}) = t+2-2k+i, \ if \ 1 \le i \le n; \ f(v_su_1) = k, \ s \in \{1,2,3,\ldots,n\}$.
We observe that, $wt(v_iv_{i+1}) = t+2+i, \ 1 \le i \le n; \ wt(v_su_1) = 2k+1, \ s \in \{1,2,3,\ldots,n\};$

$$wt(u_iu_{i+1}) = \begin{cases} 2+i, & if \ 1 \le i \le 2k-2 \\ 3+i, & if \ 2k-1 \le i \le t-1. \end{cases}$$

From the above three cases the weights of the edges of $T(n,t)$ under the labeling f constitute the set $\{3,4,5,\ldots,n+t+2\}$ and the function f is a mapping from $V(T(n,t)) \cup E(T(n,t))$ into $\{1,2,3,\ldots,k\}$. The total labeling f has the required properties of an edge irregular total labeling. Therefore we have $tes(T(n,t)) \le k$. This completes the proof. □

An edge irregular total labeling of $T(8,1)$ and $T(8,4)$ are given in Figure 1 and Figure 2 respectively.

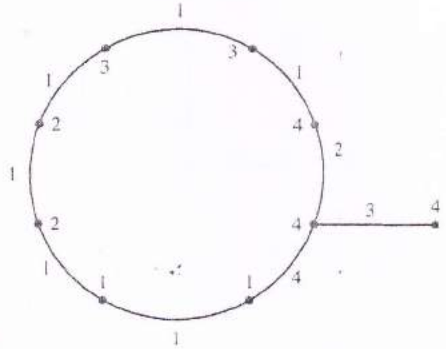

Figure 1: $tes(T(8,1)) = 4$

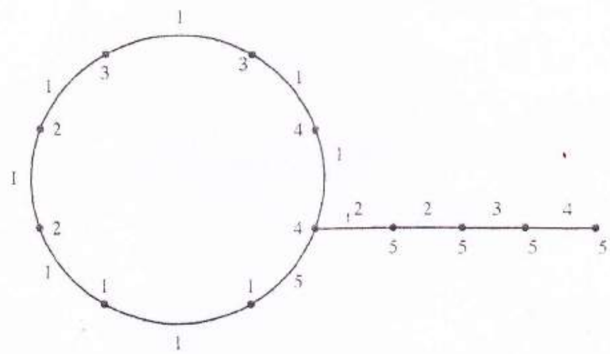

Figure 2: $tes(T(8,4)) = 5$

Lemma 2.2. $tes(C_n \ominus P_1) = \left\lceil \frac{2(n+1)}{3} \right\rceil$, *for* $n \geq 3$.

Proof. We define a total labeling f by considering the following four cases:
Case (i): $n = 3$.
$f(p_1^1) = f(p_1^2) = f(p_1^3) = 3$, $f(v_1) = f(v_2) = 1$, $f(v_3) = 2$, $f(p_1^1 v_1) = 2$,
$f(p_1^2 v_2) = f(p_1^3 v_3) = 3$, $f(v_1 v_2) = f(v_2 v_3) = 1$, $f(v_1 v_3) = 2$.
Case (ii): n=4.
$f(p_1^1) = f(p_1^2) = f(p_1^3) = f(p_1^4) = 4$, $f(v_1) = f(v_2) = 1$, $f(v_3) = 2$, $f(v_4) = 3$,
$f(p_1^1 v_1) = f(v_1 v_2) = f(v_1 v_4) = f(v_2 v_3) = 1$, $f(p_1^2 v_2) = 2$, $f(p_1^3 v_3) = f(p_1^4 v_4) = f(v_3 v_4) = 3$.
Case (iii): n=5.
$f(p_1^1) = f(p_1^2) = f(p_1^3) = f(p_1^4) = f(p_1^5) = f(v_4) = 4$, $f(v_1) = f(v_2) = 1$, $f(v_3) = 2$, $f(v_5) = 3$,
$f(p_1^1 v_1) = f(v_1 v_2) = f(v_1 v_5) = f(v_2 v_3) = 1$, $f(p_1^2 v_2) = f(p_1^3 v_3) = f(p_1^5 v_5) = 2$,
$f(p_1^4 v_4) = f(v_3 v_4) = f(v_4 v_5) = 4$.

Case (iv): $n \geq 6$.
Let $k = \left\lceil \frac{2n+2}{3} \right\rceil$
$f(p_1^i) = k,\ 1 \leq i \leq n,$

$$f(v_i) = \begin{cases} 1, & if\ 1 \leq i \leq k-2 \\ k-2, & if\ i = k-1,\ i = n \\ k, & if\ k \leq i \leq n-1; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} i, & if\ 1 \leq i \leq k-3 \\ 1, & if\ i = k-2 \\ 5, & if\ i = k-1 \\ 2n+2-k-i, & if\ k \leq i \leq n-2 \\ 4, & if\ i = n-1 \\ 2, & if\ i = n; \end{cases}$$

$$f(v_ip_1^i) = \begin{cases} i, & if\ 1 \leq i \leq k-2 \\ 2, & if\ i = k-1 \\ i-k+4, & if\ k \leq i \leq n-1 \\ 3, & if\ i = n. \end{cases}$$

We observe that,

$$wt(v_iv_{i+1}) = \begin{cases} i+2, & if\ 1 \leq i \leq k-2 \\ k+1, & if\ i = n \\ 2k+2, & if\ i = n-1 \\ 2k+3, & if\ i = k-1 \\ 2n+2+k-i, & if\ k \leq i \leq n-2; \end{cases}$$

$$wt(v_ip_1^i) = \begin{cases} k+1+i, & if\ 1 \leq i \leq k-1 \\ 2k+1, & if\ i = n \\ i+k+4 & if\ k \leq i \leq n-1. \end{cases}$$

From the above four cases the weights of the edges of $C_n \ominus P_1$ under the labeling f constitute the set $\{3, 4, 5, \ldots, 2n+2\}$ and the function f is a mapping from $V(C_n \ominus P_1) \cup E(C_n \ominus P_1)$ into $\{1, 2, 3, \ldots, k\}$. The total labeling f has the required properties of an edge irregular total labeling. Therefore we have $tes(C_n \ominus P_1) \leq k$. However by (1) $tes(C_n \ominus P_1) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{2n+2}{3} \right\rceil = k$, that is $tes(C_n \ominus P_1) \geq k$. This completes the proof. □

Figure 3 shows an edge irregular total labeling of $C_{10} \ominus P_1$.

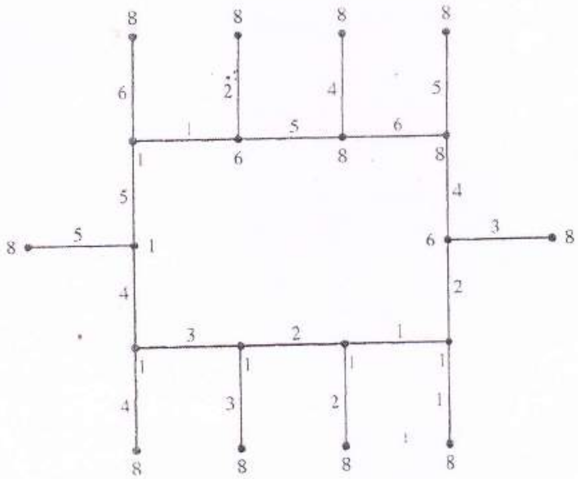

Figure 3: $\mathrm{tes}(C_{10} \ominus P_1) = 8$

Theorem 2.3. $tes(C_n \ominus P_m) = \left\lceil \frac{n(m+1)+2}{3} \right\rceil$ *for* $m \geq 2, n \geq 3$.

Proof. Let $k = \left\lceil \frac{n(m+1)+2}{3} \right\rceil$. Then from (1) it follows that, $tes(C_n \ominus P_m) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{n(m+1)+2}{3} \right\rceil = k$; that is $tes(C_n \ominus P_m) \geq k$. To prove the reverse inequality, we define a function f from $V \cup E$ into $\{1, 2, 3, \ldots, k\}$ by considering the following three cases.

Case (i) $m = 2$, $n \geq 3$.
$f(p_1^i) = 1$, $f(p_2^i) = i$ $1 \leq i \leq n$.
$f(v_i) = k$, $1 \leq i \leq n$; $f(p_1^i p_2^i) = 1, 1 \leq i \leq n$; $f(p_2^i v_i) = 1$, $1 \leq i \leq n$
$f(v_i v_{i+1}) = i$, $1 \leq i \leq n$.
Case (ii) $m = 3$, $n \geq 3$.
$f(p_1^i) = 1$, $f(p_2^i) = i$, $1 \leq i \leq n$, $f(p_3^i) = \left\lceil \frac{n}{2} \right\rceil + 1$, $1 \leq i \leq n$;
$f(v_i) = k$, $1 \leq i \leq n$; $f(p_1^i p_2^i) = 1, 1 \leq i \leq n$; $f(p_2^i p_3^i) = n + 1 - \left\lceil \frac{n}{2} \right\rceil$, $1 \leq i \leq n$;
$f(v_i p_3^i) = 2n + 1 - k - \left\lceil \frac{n}{2} \right\rceil + i$, $1 \leq i \leq n$; $f(v_i v_{i+1}) = 3n + 2 - 2k + i$, $1 \leq i \leq n$.
Case (iii) $m \geq 4$, $n \geq 3$.
$f(p_1^i) = 1$, $f(p_2^i) = 1$, $1 \leq i \leq n$,
$f(v_i) = k$, $1 \leq i \leq n$; $f(p_1^i p_2^i) = i, 1 \leq i \leq n$; $f(p_2^i p_3^i) = 1 + i$, $1 \leq i \leq n$;
$f(v_i v_{i+1}) = nm + 2(1 - k) + i$, $1 \leq i \leq n$.

For $1 \leq i \leq n$, $3 \leq j \leq m$ such that

$$f(p_j^i) = \begin{cases} \left\lfloor \frac{j-1}{2} \right\rfloor n, & if \ \left\lfloor \frac{j-1}{2} \right\rfloor n \leq k \\ k, & if \ \left\lfloor \frac{j-1}{2} \right\rfloor n > k. \end{cases}$$

For $1 \leq i \leq n$, $3 \leq j \leq m-1$ such that

$$f(p_j^i p_{j+1}^i) = \begin{cases} (j-1)n - \left\lfloor \frac{j-1}{2} \right\rfloor n - \left\lfloor \frac{j}{2} \right\rfloor n + 2 + i, & if \ \left\lfloor \frac{j-1}{2} \right\rfloor n \leq k \\ (j-1)n + 2(1-k) + i, & if \ \left\lfloor \frac{j-1}{2} \right\rfloor n > k. \end{cases}$$

$$f(p_m^i v_i) = \begin{cases} (m-1)n + 2 - k - \left\lfloor \frac{m-1}{2} \right\rfloor n + i, & if \ \left\lfloor \frac{m-1}{2} \right\rfloor n \leq k \\ (m-1)n + 2(1-k) + i, & if, \ \left\lfloor \frac{m-1}{2} \right\rfloor n > k. \end{cases}$$

We observe that, $wt(v_i v_{i+1}) = nm + 2 + i$, $1 \leq i \leq n$,
$wt(p_j^i p_{j+1}^i) = (j-1)n + 2 + i$, $1 \leq i \leq n$, $1 \leq j \leq m-1$,
$wt(p_m^i v_i) = n(m-1) + 2 + i$, $1 \leq i \leq n$.
From the above three cases the weights of the edges of $(C_n \ominus P_m)$ under the labeling f constitute the set $\{3, 4, 5, \ldots, n(m+1)+2\}$ and the function f is a mapping from $V(C_n \ominus P_m) \cup E(C_n \ominus P_m)$ into $\{1, 2, 3, \ldots, k\}$. The total labeling f has the required properties of an edge irregular total labeling. Therefore we have $tes(C_n \ominus P_m) \leq k$. This completes the proof. □

An edge irregular total labeling of $C_3 \ominus P_3$ is given in Figure 4.

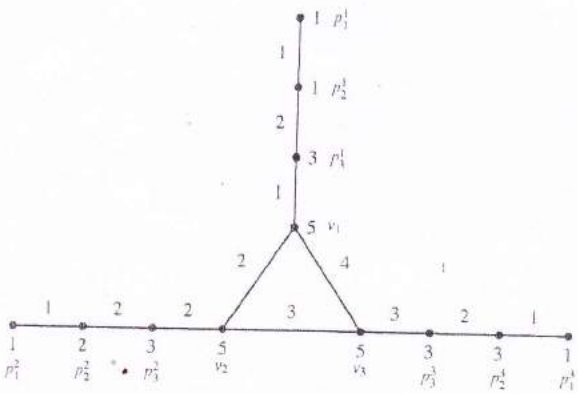

Figure 4: Armed crown graph $(C_3 \ominus P_3)$ with $tes(C_3 \ominus P_3) = 5$

Theorem 2.4. $tes(spl(C_n)) = 1 + n$, $n \geq 3$

Proof. Let $spl(C_n)$ be the split graph of C_n with $V(spl(C_n)) = \{v_i, u_i, 1 \leq i \leq n\}$ and $E(spl(C_n)) = \{v_i v_{i+1}, u_i v_{i+1}, v_i u_{i+1}, 1 \leq i \leq n\}$ with indices taken modulo n.

We define a total labeling f by considering the following two cases.
Case (i): n is even.
For $1 \leq i \leq n$;

$$f(v_i) = \begin{cases} 1+n, & if \ i \ is \ odd \\ \frac{i}{2}, & if \ i \ is \ even; \end{cases}$$

$$f(u_i) = \begin{cases} \frac{1+i}{2}, & if \ i \ is \ odd \\ 1+n, & if \ i \ is \ even; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} \frac{1+i}{2}, & if \ i \ is \ odd \\ \frac{2+i}{2}, & if \ i \ is \ even; \end{cases}$$

$$f(u_iv_{i+1}) = \begin{cases} 1, & if \ i \ is \ odd \\ i, & if \ i \ is \ even; \end{cases}$$

$$f(v_iu_{i+1}) = \begin{cases} i, & if \ i \ is \ odd \\ 1, & if \ i \ is \ even, \ 2 \leq i \leq n-2 \\ \frac{n+2}{2}, & if \ i = n. \end{cases}$$

We observe that, for $1 \leq i \leq n$,

$$wt(u_iv_{i+1}) = \begin{cases} 2+i, & if \ i \ is \ odd \\ 2n+2+i, & if \ i \ is \ even; \end{cases}$$

$$wt(v_iu_{i+1}) = \begin{cases} 2n+2+i, & if \ i \ is \ odd \\ 2+i, & if \ i \ is \ even; \end{cases}$$

$wt(v_iv_{i+1}) = n+2+i$.
Case (ii): n is odd.
For $1 \leq i \leq n$;

$$f(v_i) = \begin{cases} 1+n, & if \ i \ is \ odd, 1 \leq i \leq n-2 \\ n, & if \ i = n \\ \frac{i}{2}, & if \ i \ is \ even; \end{cases}$$

$$f(u_i) = \begin{cases} \frac{1+i}{2}, & if \ i \ is \ odd \\ 1+n, & if \ i \ is \ even; \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} \frac{1+i}{2}, & if \ i \ is \ odd, 1 \leq i \leq n-2 \\ 1, & if \ i = n \\ \frac{n+3}{2}, & if \ i = n-1 \\ \frac{2+i}{2}, & if \ i \ is \ even, 2 \leq i \leq n-3; \end{cases}$$

$$f(u_iv_{i+1}) = \begin{cases} 1, & if \ i \ is \ odd, 1 \le i \le n-2 \\ 1+n, & if \ i = n, \ n-1 \\ i, & if \ i \ is \ even, i < \left\lceil \frac{n}{2} \right\rceil \\ 1+i, & if \ i \ is \ even, i \ge \left\lceil \frac{n}{2} \right\rceil, \ 2 \le i \le n-3; \end{cases}$$

$$f(v_iu_{i+1}) = \begin{cases} i, & if \ i \ is \ odd, \ i < \left\lceil \frac{n}{2} \right\rceil \\ 1+i, & if \ i \ is \ odd, i \ge \left\lceil \frac{n}{2} \right\rceil, 1 \le i \le n-2 \\ 1, & if \ i = n \\ 1, & if \ i \ is \ even, 2 \le i \le n-1. \end{cases}$$

We observe that,

$$wt(u_iv_{i+1}) = \begin{cases} 2+i, & if \ i \ is \ odd, \ 1 \le i \le n-2 \\ \frac{5n+5}{2}, & if \ i = n \\ 2n+2+i, & if \ i \ is \ even, \ i < \left\lceil \frac{n}{2} \right\rceil \\ 2n+3+i, & if \ i \ is \ even, \ i \ge \left\lceil \frac{n}{2} \right\rceil, 2 \le i \le n-3 \\ 3n+2, & if \ \ i = n-1; \end{cases}$$

$$wt(v_iu_{i+1}) = \begin{cases} 2+i, & if \ i \ is \ even, \ 2 \le i \le n-1 \\ n+2, & if \ i = n \\ 2n+2+i, & if \ i \ is \ odd, \ i < \left\lceil \frac{n}{2} \right\rceil \\ 2n+3+i, & if \ i \ is \ odd, \ i \ge \left\lceil \frac{n}{2} \right\rceil, 1 \le i \le n-2; \end{cases}$$

$wt(v_iv_{i+1}) = n+2+i, \ 1 \le i \le n.$
From the above two cases, the weights of the edges of $spl(C_n)$ under the labeling f constitute the set $\{3, 4, 5, \ldots, 3n+2\}$ and the function f is a mapping from $V(spl(C_n)) \cup E(spl(C_n))$ into $\{1, 2, 3, \ldots, 1+n\}$. The total labeling f has the required properties of an edge irregular total labeling. Therefore we have $tes(spl(C_n)) \le 1+n$. However by (1), $tes(spl(C_n)) \ge \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{3n+2}{3} \right\rceil = 1+n$, that is $tes(spl(C_n)) \ge 1+n$. This completes the proof. □

Figure 5 and Figure 6 show an edge irregular total labeling of $spl(C_{10})$ and $spl(C_5)$.

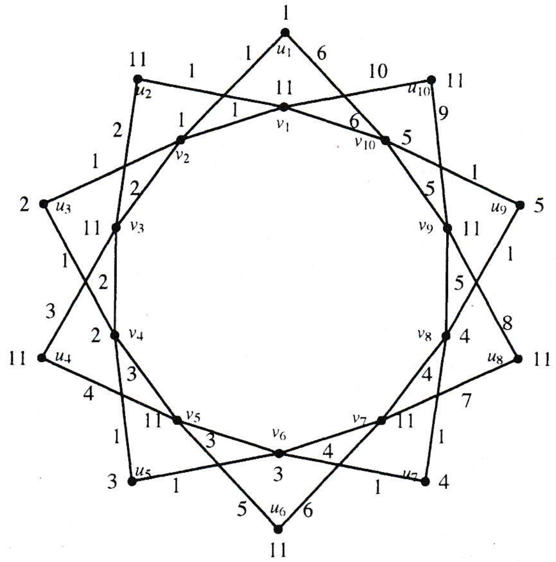

Figure 5: $tes(spl(C_{10})) = 11$

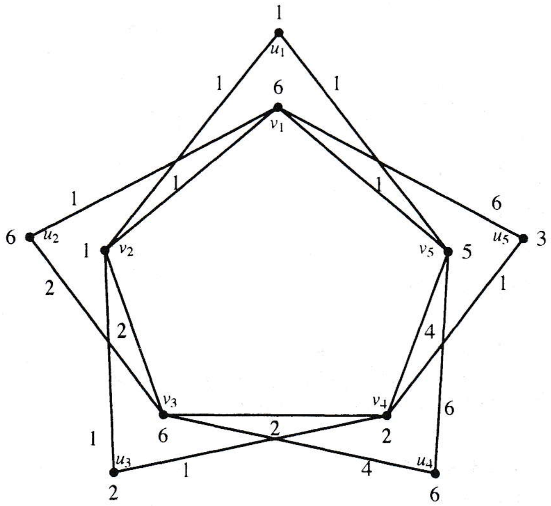

Figure 6: $tes(spl(C_5)) = 6$

Theorem 2.5. *$tes(spl(P_n)) = n$, for $n \geq 2$*

Proof. Let $V(spl(P_n)) = \{v_i,\ u_i,\ 1 \leq i \leq n\}$ and $E(spl(P_n)) = \{v_iv_{i+1},\ u_iv_{i+1},\ v_iu_{i+1},\ 1 \leq i \leq n-1\}$. We define a total labeling f by considering the following two cases.
Case (i): n is odd.

$$f(v_i) = \begin{cases} 1, & if\ i\ is\ odd, 1 \le i \le n \\ n, & if\ i\ is\ even, 2 \le i \le n-1; \end{cases}$$

$$f(u_i) = \begin{cases} n, & if\ i\ is\ odd, 1 \le i \le n \\ 1, & if\ i\ is\ even, 2 \le i \le n-1. \end{cases}$$

For $1 \le i \le n-1$, $f(v_iu_{i+1}) = \left\lceil \frac{i}{2} \right\rceil$, $f(u_iv_{i+1}) = \left\lceil \frac{n-2}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil$, $f(v_iv_{i+1}) = i$.
We observe that,

$$wt(v_iu_{i+1}) = \begin{cases} 2 + \left\lceil \frac{i}{2} \right\rceil, & if\ i\ is\ odd,\ 1 \le i \le n-2 \\ 2n + \left\lceil \frac{i}{2} \right\rceil, & if\ i\ is\ even;\ 2 \le i \le n-1; \end{cases}$$

$$wt(u_iv_{i+1}) = \begin{cases} 2n + \left\lceil \frac{n-2}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil, & if\ i\ is\ odd,\ 1 \le i \le n-2 \\ 2 + \left\lceil \frac{n-2}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil, & if\ i\ is\ even;\ 2 \le i \le n-1 \end{cases}$$

$wt(v_iv_{i+1}) = n + 1 + i,\ 1 \le i \le n-1.$
Case (ii): n is even.

$$f(v_i) = \begin{cases} 1, & if\ i\ is\ odd,\ 1 \le i \le n-1 \\ n, & if\ i\ is\ even,\ 2 \le i \le n; \end{cases}$$

$$f(u_i) = \begin{cases} n, & if\ i\ is\ odd,\ 1 \le i \le n-1 \\ 1, & if\ i\ is\ even,\ 2 \le i \le n; \end{cases}$$

For $1 \le i \le n-1$, $f(v_iu_{i+1}) = \left\lceil \frac{i}{2} \right\rceil$, $f(u_iv_{i+1}) = \left\lceil \frac{n-2}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil$;
We observe that,

$$wt(v_iu_{i+1}) = \begin{cases} 2 + \left\lceil \frac{i}{2} \right\rceil, & if\ i\ is\ odd,\ 1 \le i \le n-1 \\ 2n + \left\lceil \frac{i}{2} \right\rceil, & if\ i\ is\ even,\ 2 \le i \le n-2; \end{cases}$$

$$wt(u_iv_{i+1}) = \begin{cases} 2n + \left\lceil \frac{n-2}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil & if\ i\ is\ odd,\ 1 \le i \le n-1 \\ 2 + \left\lceil \frac{n-2}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil & if\ i\ is\ even,\ 1 \le i \le n-2; \end{cases}$$

$wt(v_iu_{i+1}) = n + 1 + i,\ 1 \le i \le n-1.$
From the above two cases, the weights of the edges of $spl(P_n)$ under the labeling f constitute the set $\{3, 4, 5, \ldots, 3n-1\}$ and the function f is a mapping from $V(spl(P_n)) \cup E(spl(P_n))$ into $\{1, 2, 3, \ldots, n\}$. The total labeling f has the required properties of an edge irregular total labeling. Therefore we have $tes(spl(P_n)) \le n$. However by (1), $tes(spl(P_n)) \ge \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{3(n-1)+2}{3} \right\rceil = n$, that is $tes(spl(P_n)) \ge n$. This completes the proof. □

Figure 7 and Figure 8 show an edge irregular total labeling of $spl(P_9)$ and

$spl(P_{10})$ respectively.

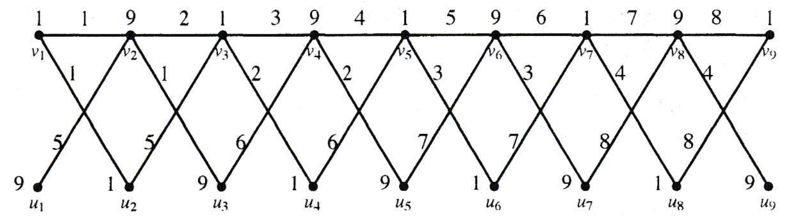

Figure 7: $tes(spl(P_9)) = 9$

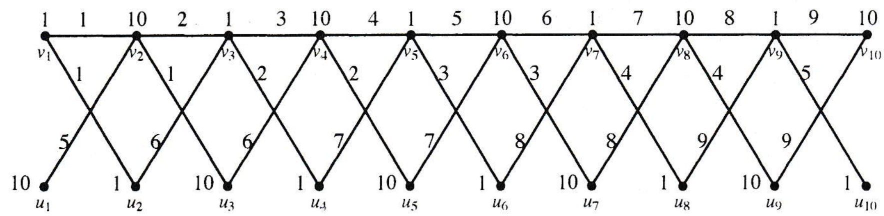

Figure 8: $tes(spl(P_{10})) = 10$

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